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Deconstructing cognitive fractures in visual-to-formal proof transitions: A diagnostic audit of proof without words tasks

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Abstract

The transition from visual intuition to formal mathematical rigor presents a persistent epistemological challenge, particularly when students must bridge the silent gaps in proofs without words. This study deconstructs cognitive fractures in the visual-to-formal proof transition among 30 mathematics education students using a Proof Without Words (PWW) task based on the Pythagorean theorem. Using a qualitative exploratory approach and a diagnostic case study design, the study employs a cross-model clinical audit that integrates Heinze and Reiss's proof competence framework with Boero's proving phases. This dual framework enables microgenetic identification of failure mechanisms during proof reconstruction. Data were collected through written work, think-aloud protocols, and task-based interviews and analyzed using a comparative diagnostic lens. The findings reveal a procedural paradox: although 80% of students demonstrated high algebraic proficiency in phase 5, success fell to 30% in phase 3 and 23% in phase 4 because of deficits in strategic and methodological knowledge. We identified three cognitive deconstruction typologies: pathological (illusion of transparency), stable (epistemic anchoring), and autonomous (ontological shift). The findings suggest that the wordless nature of PWW tasks can suppress metacognitive monitoring, producing a vertical decoupling between algebraic execution and geometric justification. This study contributes a diagnostic blueprint for developing precision scaffolding to mitigate cognitive fractures in undergraduate proof instruction.

Keywords: Cognitive Fractures, Proof Without Words, Proof Competence, Proving Phases, Diagnostic Audit

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INTRODUCTION

The maturity of mathematical proving in higher education remains a persistent pedagogical challenge, specifically in bridging the gap between visual intuition and formal rigor. The transition from visual reasoning to formal mathematical proof remains a significant issue in mathematics education. Although visual representations support students' understanding and conjecture generation, many learners experience difficulties when converting visual insights into rigorous proofs (Kontorovich et al., 2024). Students often struggle to distinguish intuitive visual arguments from formally valid mathematical reasoning (Zazkis & Rouhani, 2024). Recent reviews also confirm that proof construction and validation remain persistent challenges in mathematics learning (Lessing & Ogbonnaya, 2025). Therefore, investigating students' difficulties during visual-to-formal proof transitions remains highly relevant.

While visual representations are promoted as cognitive instruments to facilitate conceptual understanding, the process of transcoding from a mental concept image to a formal concept definition consistently reveals a profound epistemological disconnection (Posternyak & Posternyak, 2024; Wang, 2023). This phenomenon becomes critical in Proofs Without Words (PWW) tasks, where students are required to navigate the silent gaps between geometric diagrams and algebraic propositions without explicit verbal guidance (Marco et al., 2022; Polat & Demirciouglu, 2021). A significant international concern is the procedural paradox, where students exhibit proficiency in symbolic manipulation but lose logical orientation when required to explicate the deductive justifications underlying those transformations (Ayalon & Hershkowitz, 2018; Schifter & Russell, 2022). Such cognitive disruptions indicate deeply rooted fractures in students' mathematical thinking that remain under-mapped in current mainstream literature, potentially hindering the development of advanced proving literacy at the university level.

Current scholarly responses to the visual-to-formal transition often focus on the instructional efficacy of PWW or static evaluations of the final proof product (Alcock & Inglis, 2010; Hanna, 2020; U. Sholihah & Asyhar, 2023). Experts like Stylianides & Stylianides (2008) and Jaeger (2025) argue that the extreme transparency of visual diagrams often triggers an illusion of understanding, where the perceptual clarity lulls students into bypassing formal deductive verification. However, most previous studies

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are descriptive and general and fail to conduct a diagnostic audit that can chronologically track the precise cognitive fractures that occur during the navigation process (Claessen & van der Ham, 2017; Dawkins & Weber, 2017; N. Sholihah & Aini, 2023). A critical research gap remains in how internal proof components interact with specific argumentation phases when students face wordless stimuli. Without a micro-level deconstruction of these internal mechanisms, the paradox of the algebraically proficient but geometrically paralyzed student will remain a black box that is difficult to intervene in pedagogically (Koichu et al., 2007; Wahyuni et al., 2025).

To uncover the anatomy of these failures, an analytical framework is required that simultaneously captures both the structure of knowledge and the dynamics of the process. (Heinze & Reiss, 2002) proof competence model provides the theoretical foundation for identifying the availability of basic knowledge regarding definitions and theorems; strategic knowledge in selecting navigational routes; and methodological knowledge encompassing the understanding of formal proof norms and logical structures. Theoretically, these cognitive capacities must synergize with the sequential argumentation phases defined in (Boero, 1999) model. These phases, including the production of a conjecture, formulation of a statement, exploration of a strategy, construction of an argumentation chain, formal proof execution, and final validation, represent the temporal journey of a proof (Turiano et al., 2020). Integrating these two primary models provides a precision audit instrument to diagnose whether student failure is rooted in a deficit of specific knowledge components or a disruption within a particular argumentation phase when facing the high cognitive demands of a PWW stimulus (Bell, 2011; Dreyfus & Kidron, 2014; Haj-Yahya et al., 2023).

This study aims to deconstruct cognitive fractures in the visual-to-formal proof transition among 30 mathematics education students through Proof-Without-Words tasks. Specifically, the study aims to diagnose disruptions in proof competence components (Heinze & Reiss, 2002) and map cognitive fractures across each argumentation phase (Boero, 1999) while students reconstruct the Pythagorean theorem. This article's novelty lies in applying a cross-model clinical audit that integrates Heinze & Reiss's competence model with Boero's phases to reveal students' cognitive pathology profiles, often hidden behind procedural algebraic proficiency. This research focuses on uncovering systemic failure mechanisms in students' metacognitive

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monitoring units when facing visual cognitive load, ultimately contributing a new theoretical basis for developing precision scaffolding strategies to mitigate cognitive disruptions in undergraduate proof instruction. Accordingly, investigating cognitive fractures during the transition from visual representations to formal proof becomes increasingly relevant, as contemporary mathematics education continues to seek effective ways to bridge intuitive reasoning and deductive rigor (Hsu & Silver, 2026; Kontorovich et al., 2024).

METHODS

This research employed a qualitative-exploratory approach with a diagnostic case study design to perform a microgenetic deconstruction of students' proving processes (Yin, 2018). The study focused on identifying cognitive fractures during the transition from visual stimuli to formal mathematical proof. The participants were 30 undergraduate students from the Mathematics Education department who had completed fundamental courses in geometry and algebra. Participants were selected because they had sufficient prior knowledge of geometry and algebra to attempt formal proofs, making them suitable for examining cognitive fractures that emerge during the transition from visual representations to formal mathematical reasoning. The subjects were selected using a purposive sampling technique to capture a diverse range of proof-of-competence profiles when faced with wordless mathematical tasks.

The primary instrument used in this study was a PWW task based on Garfield's Pythagorean proof model. This task required students to reconstruct a formal algebraic proof from a silent geometric diagram, thereby creating a silent gap that necessitated autonomous cognitive bridging. Data were collected using think-aloud protocols and task-based interviews to uncover the students' stream of consciousness and metacognitive justifications during the proving process (Baxter et al., 2015). We identified metacognitive monitoring from transcript segments in which students explicitly checked, evaluated, questioned, or revised their reasoning while constructing proofs. These segments served as analytical units for identifying cognitive fracture points during the visual-to-formal proof transition. We recorded, transcribed, and coded all verbalizations and written scripts for deep analysis.

Data analysis followed a cross-model clinical audit procedure, integrating the frameworks of Heinze & Reiss (2002) and Boero (1999). A cognitive fracture was

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operationally defined as a moment in which a student's proving trajectory became discontinuous due to the inability to transform visual evidence into a formally justified mathematical statement. The diagnostic deconstruction was conducted through four systematic stages: (1) Mapping the students' navigational flow according to Boero's six proving phases, from conjecture production to formal validation; (2) Auditing the activation of Heinze and Reiss's three knowledge components (basic, strategic, and methodological) within each identified phase. (3) Identifying precise cognitive fracture points where the transition from visual intuition to formal rigor was disrupted; and (4) Synthesizing these findings into cognitive pathology profiles. Data validity was ensured through source triangulation and member checking to maintain the interpretive accuracy of the think-aloud transcripts and the diagnostic conclusions.

RESULT AND DISCUSSION

Micro-Chronological Diagnostic Audit

A clinical audit of the proof reconstruction process involving 30 research participants revealed a striking contrast in cognitive stability between the visual perception phase and the symbolic formalization phase. Using Boero's analytical framework and the competence components proposed by Heinze and Reiss, the findings indicate that students' proving profiles do not develop linearly or remain static. Instead, they undergo a series of disruptions at specific transitional points within the proving process. Table 1 summarizes the frequency distribution of successful performances and the identified knowledge deficits across each unit of proof navigation.

Table 1. Micro-Genetic Audit of Success Frequency and Knowledge Deficit

	Proving Phases	Success	Percentage	Knowledge Component	Status
1	Production of a Conjecture	30	100%	Basic Knowledge (Visual-Intuitive)	Stable
2	Formulation of the Statement	28	93%	Basic Knowledge (Representational)	Stable
3	Exploration of Content & Strategy	9	30%	Strategic Knowledge	Critical Fracture
4	Construction of Argumentation Chain	7	23%	Methodological Knowledge	Critical Fracture
5	Execution of Formal Proof	24	80%	Procedural Knowledge (Algebraic)	Dominant
6	Final Validation	6	20%	Methodological Knowledge	Abstraction Barrier

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The data presented in Table 1 diagnose a phenomenon defined in this study as the procedural paradox. All participants (100%) showed complete stability during phase 1 (conjecture production), where they intuitively recognized area relationships through the PWW diagram. However, this stability collapsed in phase 3 (strategy exploration), where only 30% of participants designed a logically valid navigational pathway. This sharp 70% decline indicates a deficit in strategic knowledge; students had the necessary raw materials (formulas) but lacked the navigational map needed to connect geometric components into a coherent proving strategy.

The audit further detected deeper cognitive fractures during phase 4 (argument-chain construction). Only 23% of participants could construct a deductive sequence that connected geometric facts, such as the properties of parallel sides and right angles within the trapezoid, into a valid algebraic proposition. The disruption observed in this phase was diagnosed as a weakness in methodological knowledge. Students tended to disregard the norms of formal proof and instead performed a cognitive leap, moving directly to phase 5 (formal manipulation) without establishing a rigorous deductive foundation.

Interestingly, during phase 5, the success rate increased sharply to 80%. This pattern suggests that most students had strong algebraic operational proficiency. However, this proficiency proved to be largely superficial, as it was not supported by a valid chain of arguments established in the preceding phases. This cognitive pathology culminated in phase 6 (final validation), where the success rate fell to its lowest level (20%). This obstacle is identified as an abstraction barrier. Although students could complete the equation $a^2 + b^2 = c^2$, they failed to validate that this relationship represents a universal, mathematically invariant truth independent of the specific physical diagram provided.

Micro-Genetic Deconstruction of the Argumentation Phase and Proving Competence

A qualitative–exploratory analysis of the 30 research participants revealed a complex dynamic in the visual-to-formal transition during the proving process. To better understand the internal mechanisms underlying proof construction, this study focuses its analytical deconstruction on key participants representing distinct cognitive profiles: cognitive pathology, competence stability, and cognitive autonomy.

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The Pathology of the Epistemological Shortcut

Subject S5 was selected as a representative case of the cognitive pathology pathway identified in the cognitive fracture model. This pathway describes students who interpret visual information successfully but prematurely shift to procedural manipulation without building sufficient deductive justification. The analysis of S5's proving process illustrates how this cognitive fracture develops during the transition from visual reasoning to formal proof. During phase 1 and phase 2, Subject S5 demonstrated a highly stable level of basic knowledge. Visual evidence from S5's worksheet indicates that the participant labeled variables *a*, *b*, and *c* quickly and accurately (Figure 1). In the think-aloud protocol, S5 murmured: *"This is a trapezoid made from two identical triangles and one triangle in the middle... the height is* a + b."

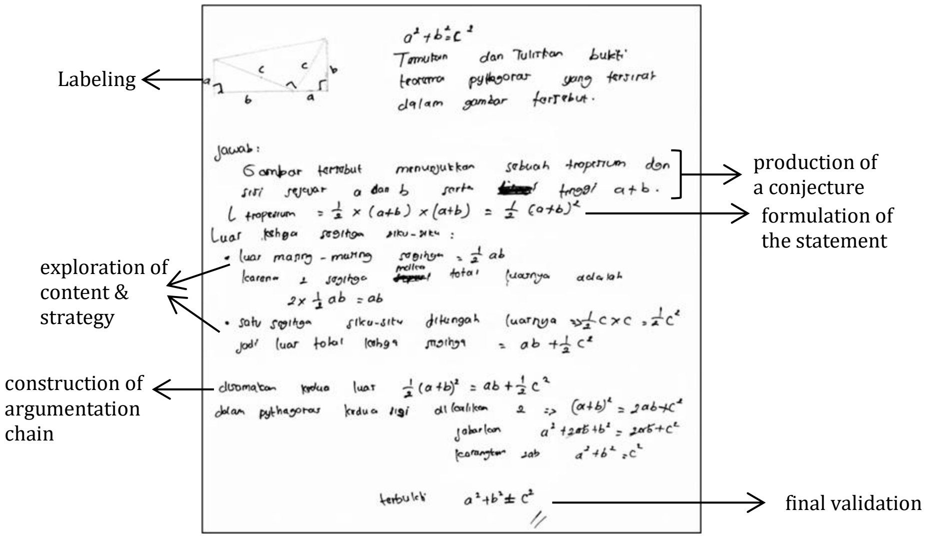
This ability indicates that at the level of initial visual perception, no cognitive disruption was detected. The students' perceptual system functioned effectively in identifying the diagram's structural elements and constructing an initial conjectural representation.

However, a sharp disruption emerged in phases 3 and 4. At this stage, S5's strategic knowledge broke down. Instead of verifying the geometric properties of the trapezoid's constituent components, S5 immediately jumped to algebraic operations. In the diagnostic interview, S5 admitted:

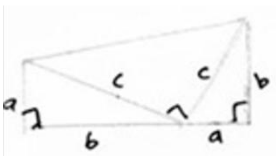
> *"The diagram is already very transparent and clear, so I do not need to check the right-angle markings again. I should move directly to algebra so I do not lose the calculation flow."*

This phenomenon reflects what can be described as an illusion of transparency, in which the PWW diagram's apparent clarity paradoxically suppresses students' deductive awareness. As a result, they bypass the crucial phase of content exploration in favor of procedural efficiency.

The phenomenon of cognitive disruption during proof reconstruction is clearly evident in the work produced by Subject S5 (Figure 1). S5 attempted to transition rapidly from visual inspection of the PWW diagram to algebraic manipulation without first constructing a coherent chain of geometric arguments. This behavior reflects what this study identifies as an epistemological shortcut, in which procedural efficiency replaces deductive justification.



Translation of Figure 1:



$a^2 + b^2 = c^2$

Discover and write down the proof of the Pythagorean theorem implied in the given figure.

Response

The figure illustrates a trapezoid with parallel sides a and b, and a height of $a + b$.

Area of the trapezoid $= \frac{1}{2} \times (a+b) \times (a+b) = \frac{1}{2}(a+b)^2$

The areas of the three right triangles are determined as follows:

- The area of each [outer] triangle is $\frac{1}{2}ab$. Since there are 2 triangles, their combined area is $2 \times \frac{1}{2}ab = ab$.
- The inner right triangle has an area of $\frac{1}{2} \times c \times c = \frac{1}{2}c^2$. Thus, the total area of the three triangles is $ab + \frac{1}{2}c^2$.

Equating both area expressions: $\frac{1}{2}(a+b)^2 = ab + \frac{1}{2}c^2$

Multiplying both sides by 2 yields: $(a+b)^2 = 2ab + c^2$

Expanding the algebraic expression: $a^2 + 2ab + b^2 = 2ab + c^2$

Subtracting $2ab$ from both sides results in: $a^2 + b^2 = c^2$

Proven: $a^2 + b^2 = c^2$

Figure 1. Subject S5's solution illustrating the epistemological shortcut phenomenon during the transition from visual inspection to formal proof construction.

The Stability of the Epistemological Anchor and Methodological Rigor

Subject S9 was selected as a representative case of the Competence Stability pathway in the cognitive fracture model. The case illustrates how the stable integration of basic, strategic, and methodological knowledge supports successful proof reconstruction from visual representations. In contrast, Subject S9, who experienced cognitive disruption, demonstrated high proof maturity through consistent synchronization across all phases of proof navigation. S9's success appears to stem from strong methodological knowledge: the visual stimulus in the PWW task was not treated as a finished fact, but as a collection of entities that needed systematic redefinition.

During phase 1 and phase 2, S9 showed high stability in basic knowledge. Visual evidence (Figure 2) from the participant's worksheet revealed numerical partitioning, where Roman numerals (II, III, and IV) were written inside each triangular component forming the trapezoidal configuration. This strategy served as an epistemological anchor, allowing the participant to secure the identity of each geometric component before moving into more complex algebraic manipulation. In the think-aloud protocol, S9 explicitly articulated this monitoring process:

> *"I numbered each part first (II, III, IV) so that I would not get confused when calculating the total area later. I need to know which one is the big triangle and which one is the triangle in the middle."*

This careful approach continued into phase 3 and phase 4. Rather than immediately applying the algebraic operation $(a + b)^2$, S9 decomposed the area of each numbered region separately. The participant explicitly identified the height of the trapezoid as (a+b), indicating a mature level of strategic knowledge for bridging the gap between the visual representation and its formal algebraic structure. During the diagnostic interview, S9 further explained:

> *"I wrote the area of each numbered part separately under the diagram, and only then did I equate it with the area of the large trapezoid. That way, I am sure no variable gets mixed up."*

This behavior indicates that S9 used the numbering of geometric regions as a mental offloading strategy, reducing the cognitive load on working memory during proof construction.

The success of S9 in phase 5 and phase 6 can therefore be understood as a logical consequence of the participant's methodological precision in the earlier phases. Because the geometric relationships had already been secured through the numbering scheme (II, III, IV), the algebraic execution proceeded without semiotic interference. S9 concluded the proof with the symbol "∴ proven," indicating a state of epistemic certainty. S9's case suggests that students who can synchronize knowledge components and proof phases in a disciplined manner show greater resistance to the transparency illusion. This phenomenon often traps students when working with proof tasks presented without words.

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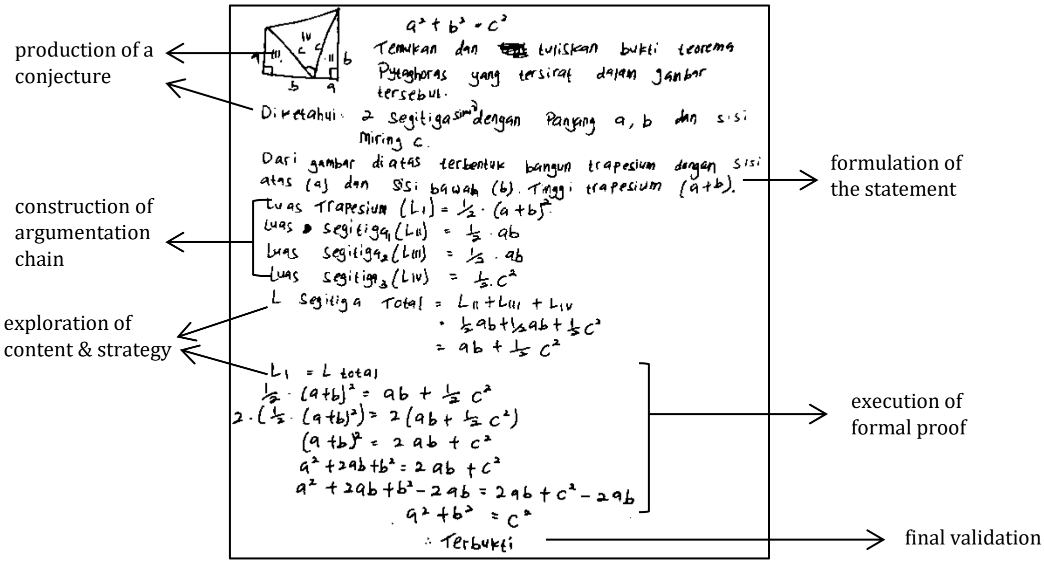
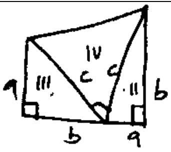


Translation of Figure 2:



$a^2 + b^2 = c^2$

Discover and write down the proof of the Pythagorean theorem implied in the given figure.

Given: Two right triangles with side lengths a, b and hypotenuse c.
From the figure above, a trapezoid is formed with a top base (a) and a bottom base (b). The height of the trapezoid is $(a + b)$.

Area of the Trapezoid $(L_I) = \frac{1}{2} \cdot (a + b)^2$

Area of triangle1 $(L_{II}) = \frac{1}{2} \cdot ab$

Area of triangle2 $(L_{III}) = \frac{1}{2} \cdot ab$

Area of triangle3 $(L_{IV}) = \frac{1}{2} \cdot c^2$

$$\begin{aligned}
\text{Total Area of Triangles} &= L_{II} + L_{III} + L_{IV} \\
&= \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2 \\
&= ab + \frac{1}{2}c^2 \\
L_I &= \text{Total Area} \\
\frac{1}{2} \cdot (a+b)^2 &= ab + \frac{1}{2}c^2 \\
2 \cdot \left(\frac{1}{2} \cdot (a+b)^2\right) &= 2\left(ab + \frac{1}{2}c^2\right) \\
(a+b)^2 &= 2ab + c^2 \\
a^2 + 2ab + b^2 &= 2ab + c^2 \\
a^2 + 2ab + b^2 - 2ab &= 2ab + c^2 - 2ab \\
a^2 + b^2 &= c^2
\end{aligned}$$

∴ Proven

Figure 2. Subject S9's solution in the PWW task showing the numbering of geometric parts

Cognitive Autonomy through System Redefinition

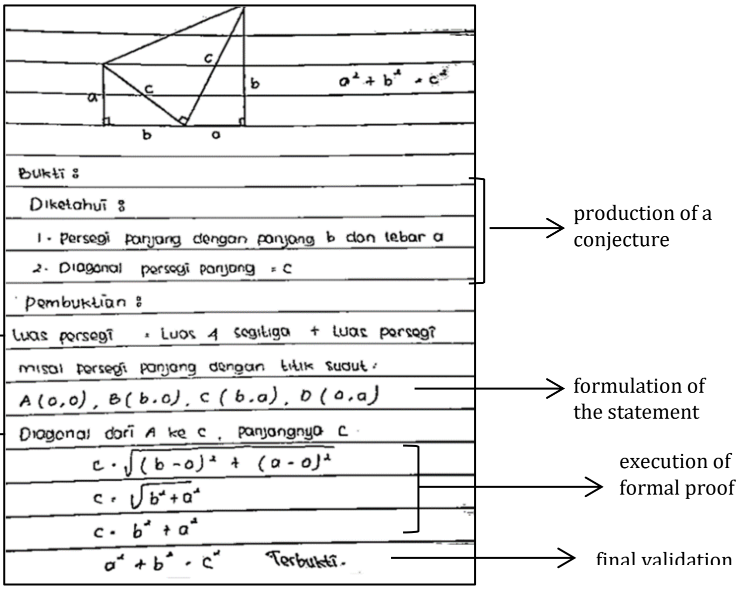
Subject S6 was selected as a representative case of the cognitive autonomy pathway in the cognitive fracture model. The case illustrates how advanced strategic and methodological knowledge enables students to flexibly reconstruct proof structures and successfully transcend the constraints of the original visual representation. Subject S6 demonstrated the most sophisticated proof profile in this study, characterized by advanced strategic knowledge that enabled an ontological shift when encountering ambiguity in the PWW stimulus. During phase 1 and phase 2, S6 showed stable basic knowledge (visual–intuitive). As shown in Figure 3, the participant transformed the Garfield trapezoid diagram into a rectangle with length (b), width (a), and diagonal (c). This reconstruction indicates an early mental simplification of the visual configuration.

The turning point emerged in phase 3. Through think-aloud reasoning, S6 initially proposed a conventional area-based approach ("*Area of the square = area of four triangles + area of the square?*"). However, through metacognitive monitoring, the participant abandoned this route and redefined the system using a Cartesian coordinate framework. In the interview transcript, S6 explained that working with coordinates felt "*safer*" than relying on the visually fragmented structure of the original diagram.

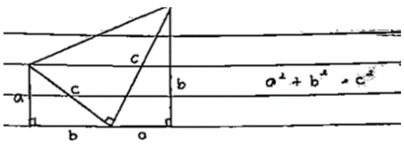
During phase 4 and phase 5, S6 demonstrated strong methodological knowledge by constructing a concise deductive chain. S6 defined the diagonal c as the distance between points $A(0,0)$ and $C(b, a)$. By applying the distance formula $c = \sqrt{(b-0)^2 + (a-0)^2}$, S6 derived $c^2 = a^2 + b^2$ with minimal algebraic manipulation and without symbolic interference.

Finally, in phase 6 (final validation), S6 concluded the proof with the statement "*proven*," indicating epistemic certainty. This case illustrates how cognitive autonomy allows learners to transcend the diagram's visual constraints and reconstruct the theorem through a formally defined analytical system.

Based on S6's written work in Figure 3, a detailed diagnostic analysis confirms that this participant is a clear example of an autonomous–transformative proof profile. Interestingly, although S6 initially reproduced the Garfield trapezoid diagram, the participant consciously abandoned the conventional area-based strategy and instead redefined the system through analytic geometry.



Translation of Figure 3:



Proof:
Given:
1. A rectangle with length b and width a
2. The diagonal of the rectangle = c
Proofing:
Area of the square = Area of 4 triangles + Area of the square
Let a rectangle have the following vertices:
$A(0,0), B(b,0), C(b,a), D(0,a)$
The diagonal from A to C has a length of c.
$c = \sqrt{(b-0)^2 + (a-0)^2}$
$c = \sqrt{b^2 + a^2}$
$c = b^2 + a^2$
$a^2 + b^2 = c^2$ Proven.

Figure 3. Subject S6's work illustrating the transformation of the PWW diagram into a Cartesian coordinate representation.

The findings reveal a complex cognitive structure underlying the transition from visual to formal proof, manifested in three distinct patterns of cognitive deconstruction. This phenomenon, conceptualized as a procedural paradox, is interpreted through the integration of the competence model of (Heinze & Reiss, 2002) and the proving phase framework proposed by (Boero, 1999). The predominance of failure in phase 3 and phase 4, particularly in case S5, indicates an epistemological shortcut. Although participants demonstrated stable basic knowledge, breakdowns occurred in their strategic and

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methodological knowledge (Sporn et al., 2025).

Consistent with prior studies, the visual clarity of PWW tasks may generate an illusion of transparency, leading students to develop premature epistemic certainty and bypass deductive verification (Carpenter & Sanchez, 2025; Mudaly, 2013; Rarichan et al., 2024). Consequently, learners often skip intermediate reasoning stages and move directly to phase 5 without constructing a coherent argumentative chain. From a cognitive perspective, this pattern suggests that strong visual affordances may suppress metacognitive monitoring, resulting in procedural algebraic competence without preserving geometric meaning. Such behavior reflects a vertical decoupling between algebraic proficiency and geometric reasoning (Ayalon et al., 2017; Hsu & Silver, 2026).

In contrast, participant S9's successful trajectory highlights the importance of what this study conceptualizes as epistemological anchors. Labeling and verbal explanatory narration functioned as cognitive supports that stabilized visual representations and maintained coherence between geometric structures and symbolic expressions. This finding aligns with Cognitive Load Theory, which posits that external representational supports such as labeling reduce working-memory demands through mental offloading (Paas & van Merriënboer, 2020; Sweller et al., 2019).

By establishing clear geometric identities during the strategy exploration phase, participants reinforced their methodological knowledge, ensuring that the subsequent transcodification into algebraic expressions during formalization remained consistent with their internal representations. The observed stability suggests that proving maturity extends beyond content mastery to include metacognitive regulation of reasoning processes. Rather than treating the diagram as self-explanatory, successful participants actively imposed a formal interpretive structure on the visual stimulus (Dawkins & Weber, 2017; Stylianides & Stylianides, 2008).

The study's most significant contribution is identifying autonomous cognitive deconstruction, exemplified by participant S6. In this pattern, learners disengaged from the constraints of the original Garfield diagram and reconstructed the mathematical relationships within an alternative representational system. In one case, the trapezoidal configuration was reformulated within a Cartesian coordinate framework, transforming spatial relations into analytic expressions of coordinate geometry. This transformation reflects an ontological shift in which the mathematical object is reconceptualized from a

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visual–spatial configuration into a generalized analytical structure (Helenius & Ahl, 2025; Koichu et al., 2007).

The successful completion of the final validation phase suggests that mathematical certainty emerges when learners transcend the initial visual stimulus (Stylianides et al., 2024). From this perspective, the PWW diagram functions not merely as a pedagogical scaffold but as a cognitive trigger that learners must ultimately surpass to reveal the theorem's generalized logical structure.

Overall, the diagnostic analysis involving thirty participants indicates that failures in proof construction often arise from systemic breakdowns in the metacognitive control layer of reasoning rather than from deficiencies in mathematical content knowledge (Inglis & Mejía-Ramos, 2009; Weber, 2004). Although students may know the required algebraic procedures, they often lack the strategic coordination needed to construct coherent deductive arguments.

The transition from visual reasoning to formal proof therefore does not occur linearly. However, it involves a critical transition point during the strategy exploration phase, where cognitive regulation determines the trajectory of proof reconstruction. To synthesize these findings, this study proposes the cognitive fracture model of PWW reconstruction, which illustrates three pathways—pathological, stable, and autonomous deconstruction. By integrating Boero's (1999) proving phases with Heinze and Reiss's (2002) competence components, the model explains the mechanisms that shape the success or failure of proof reconstruction (see Figure 4).

Beyond its theoretical contribution, the Cognitive Fracture Model offers a practical diagnostic framework for mathematics lecturers. The model identifies the strategy exploration phase (Boero Phase 3) as a critical transition point where students' cognitive regulation determines the trajectory of proof reconstruction. By examining students' written work, verbal explanations, and proof strategies, lecturers can diagnose whether learners follow a pathological, stable, or autonomous pathway. Students who move prematurely from visual inspection to symbolic manipulation without establishing geometric relationships may take an epistemological shortcut, leading to fragmented arguments and unsuccessful proof construction. In contrast, students who use epistemological anchors to organize and verify visual information are more likely to construct coherent deductive arguments. Meanwhile, students who redefine the problem

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through alternative mathematical systems, such as coordinate geometry or similarity transformations, demonstrate cognitive autonomy. Consequently, the model can be used not only to identify where proof reconstruction fails or succeeds, but also to guide targeted instructional interventions tailored to different proving profiles.

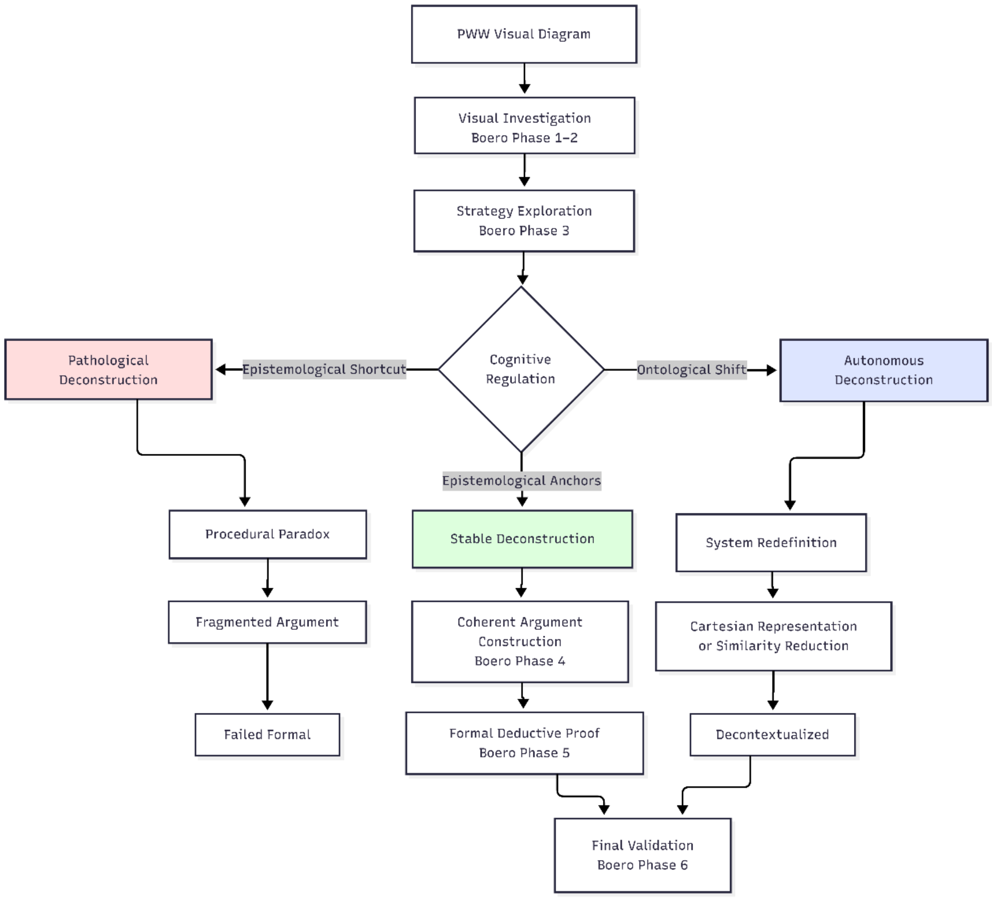

Figure 4. Cognitive Fracture Model of PWW Reconstruction.

CONCLUSION

This study presents a diagnostic analysis of cognitive fractures emerging in the transition from visual reasoning to formal proof among preservice mathematics teachers engaged with Proof Without Words (PWW) tasks. The findings show that failures in proof reconstruction are primarily linked to disruptions in strategy exploration and argument construction rather than to deficiencies in algebraic content knowledge. The study identified a procedural paradox: many students demonstrated strong algebraic operational skills yet struggled to transform visual representations into coherent deductive arguments. This difficulty is linked to a transparency illusion, where the apparent clarity of PWW diagrams suppresses metacognitive monitoring and diminishes the perceived need for formal verification. The study further identifies three cognitive trajectories—pathological, stable, and autonomous deconstruction—representing different forms of cognitive regulation during proof reconstruction. Building on these patterns, the study proposes the cognitive fracture model of PWW reconstruction, which explains how variations in strategic, methodological, and metacognitive coordination shape students' proof trajectories. Pedagogically, the findings highlight the importance of diagnostic-based scaffolding strategies that strengthen methodological knowledge to bridge the gap between visual intuition and formal deductive reasoning. Future research should examine the applicability of the cognitive fracture model across different mathematical proof domains and educational levels. Further studies may also develop the model into a practical diagnostic instrument to identify and address students' cognitive fractures during proof construction.

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AUTHOR CONTRIBUTIONS

The contributions of the four authors are described as follows:

First Author (Corresponding Author): Conceptualization, methodology, formal analysis, and writing – original draft preparation.

Second Author: Data curation, validation, and software development for micro-genetic analysis tools.

Third Author: Investigation (implementation of think-aloud protocols), visualization (mapping of proof phases), and resources.

Fourth Author: Writing – review and editing, supervision, and project administration.

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