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On the RACN of the comb product of the cycle C_3 with path P_n and broom $Br_{n,m}$

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Abstract

The combination of rainbow coloring and anti-magic labeling is known as Rainbow Antimagic Coloring (RAC). The Rainbow Antimagic Connection Number (RACN) of a graph G is the smallest number of colors induced by all edge weights under an antimagic labeling, symbolized as $rac(G)$ A graph is said to have rainbow antimagic connectivity if for every pir of vetices $x \in V(G)$, there exits a rainbow antimagic path**,** wherin all edge weights along the path are distinct. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A bijective function f from $V(G)$ to $\{1,2,\dots,|V(G)|\}$ is applied, wherein the weight of the edge $uv \in E(G)$ is defined as $w(uv)$ under f which is obtained from $w(xv) = f(x) + f(v)$. A rainbow path $x - v$ is a path in a vertex-labeled graph G if for any two edges $xv, x'v' \in E(P)$ the path satisfies $w(xv) \neq w(x'v')$. If there is a rainbow $x - v$ path P for every two vertices $x, v \in V(G)$ then the function f is called a rainbow antimagic labeling of G. A graph G we say has an RAC, if we assign each edge xv with an edge weight color $w(xv)$. In this paper, we present the RACN of the comb product of cycle C_3 with path P_n and broom $Br_{n,m}$ symbolized by $C_3 \rhd P_n$ and $C_3 \rhd Br_{n,m}$. A comb operation on a graph G, symbolized as $G \rhd H$, is a graph product wherein each vertex of G is replaced by a copy of H, maintaining the structure of G. This operation helps construct new classes of graphs with specific connectivity and labeling properties.

Keywords: Antimagic Labeling, RACN, Comb Product of Graphs, Rainbow Coloring

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INTRODUCTION

A graph G is a nonempty finite set $V(G)$, with the elements referred to as vertices, along with a set $E(G)$, consisting of edges, which are unordered pairs of distinct elements

from $V(G)$. The graph theory definition applied here can be found in Chartrand et al. (2016).

Graph theory is crucial in modern computational and mathematical modeling, particularly in network security, bioinformatics, and optimization problems. One significant aspect of graph theory is graph coloring, which has been extensively studied due to its applications in network routing, scheduling, and cryptography. A specialized form of graph coloring known as Rainbow Antimagic Coloring (RAC) has gained attention because of its secure communication and data encryption implications. However, despite its theoretical importance, the study of Rainbow Antimagic Connection Numbers (RACN) in graph operations remains underexplored. RACN of a graph G does all edge weight induce the smallest number of colors under an antimagic labeling, symbolized as $rac(G)$.

Understanding the RACN in complex graph structures, such as the comb product of graphs, is essential because combination operations on graphs often reveal new patterns in connectivity and edge-labeling schemes. The cycle graph C_3, path graph P_n, and broom graph $Br_{n,m}$ are fundamental structures in network topology, wherein efficient connectivity and labeling schemes can significantly impact theoretical and applied mathematical studies. This research fills an existing gap by providing precise values of RACN for the comb product of these graphs, contributing to a deeper understanding of their structural properties.

A graph $G = (V, E)$ is made up of a set of vertices $V(G)$ and a set of edges $E(G)$. The investigation of graph connectivity through rainbow coloring and antimagic labeling has attracted considerable interest because of its relevance to network security, data communication, and cryptography.

A rainbow path in a graph is a path wherein all edges are assigned distinct colors. A graph is considered rainbow connected if at least one rainbow path exists between every pair of vertices. Antimagic labeling is a technique wherein each vertex or edge is assigned unique weights to satisfy certain mathematical constraints. The RACN of a graph G is the smallest number of colors required to ensure that every pair of vertices in G is connected by at least one rainbow antimagic path, wherein each edge weight follows an antimagic labeling rule. This concept enhances traditional rainbow connection by incorporating edge-weight uniqueness into the connectivity criteria. In this paper, the graph operation we use is the comb product of graphs. Suppose G_1 and G_2 are connected

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graphs and v is a vertex in G_2 which is called the graft vertex. The comb product graph of graphs G_1 and G_2 is symbolized as $G_1 \rhd G_2$ is the graph formed by taking one copy of G_1 and $V(G_1)$ copies of G_2, then grafting the vertex v from the i-th copy of graph G_2 to the i-th vertex of graph G_1, as described by Chartrand et al. (2016). The graphs used are the comb product graph of the cycle C_3 with the path P_n and the broom graph $Br_{n,m}$ which is symbolized by $C_3 \rhd P_n$, with the grafted vertices being vertex x and v wherein $deg(x) = 1$ and $d(v, x) = 1$ and $C_3 \rhd Br_{n,m}$ with the grafted vertices being vertex x and v in a path from broom $Br_{n,m}$ wherein $deg(x) = 1$ and $d(v, x) = 1$.

This paper provides an in-depth exploration of the results obtained from the study of Rainbow Antimagic Coloring (RAC). This concept combines the principles of rainbow coloring and antimagic labeling in graph theory. The concept of rainbow connection, which underpins RAC, was initially introduced by Chartrand et al. (2008) and has since become a key area of interest in graph connectivity. Building on this foundation, significant advancements have been made in Rainbow Coloring, with pivotal results published in works by Li et al. (2012) and Li et al. (2017), expanding our understanding of how rainbow coloring can be applied to various graphs.

In addition to the general concept of Rainbow Coloring, other important variations, such as Rainbow Vertex Coloring (RVC) and Total Rainbow Coloring, have been developed. RVC, introduced by Krivelevich and Yuster (2010), addresses the challenge of vertex coloring in rainbow contexts and has been further investigated in works by Simamora and Salman (2010) and Li and Shi (2013). Total Rainbow Coloring, which considers coloring both vertices and edges in a graph, has been explored in detail in studies by Sun (2015) and Hasan et al. (2018), providing a deeper understanding of how this concept can be extended to more complex graphs.

The foundation of graph labeling in general, which includes antimagic labeling, is well-documented in Wallis et al. (2001), while Hartsfield and Ringel (1990) were the first to formally define antimagic labeling as a method where the sum of the labels of adjacent vertices or edges is distinct. Over the years, several authors have contributed to expanding this field, with notable results from Bača et al. (2000, 2004, 2007, 2009), who investigated different variations of antimagic labeling and its applications. In addition, the works of Dafik et al. (2004), Chang et al. (2016), and Cranston (2009) have enriched the understanding of how antimagic labeling can be applied to various graph structures

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and problems.

Arumugam et al. (2017) contributed significantly to this field by combining antimagic labeling with graph coloring to introduce the concept of local antimagic coloring. This development has opened up new avenues for research, particularly in understanding how local properties of a graph can influence global coloring schemes.

Furthermore, this research has practical applications in forecasting studies and has been integrated into recent works in applied mathematics and computational modeling. Studies such as those conducted by Rohmah and Mufaidah (2024), Yudistira et al. (2024), Jannah and Hamid (2024), Surur et al. (2024), and Nasution and Granita (2024) demonstrate the versatility of RAC and its potential for solving complex problems in areas such as data transmission, network security, and cryptographic applications.

According to the definition by Arrumugam and Dafik et al., combined rainbow coloring and antimagic labeling into a study of RAC (Dafik et al., 2020). Let G be a connected graph and the bijective function f from $V(G)$ to $\{1,2,\dots,|V(G)|\}$. The weight of the edge $uv \in E(G)$ is symbolized by $w(uv)$ under f which is obtained from $w(xv) = f(x) + f(v)$. A rainbow path $x - v$ is a path P in a vertex-labeled graph G if for any two edges $xv, x'v' \in E(P)$ the path satisfies $w(xv) \neq w(x'v')$. If there is a rainbow $x - v$ path for every two vertices $x, v \in V(G)$ then the function f is called a rainbow antimagic labeling of G. A graph G we say has an RAC, if we assign each edge xv with an edge weight color $w(xv)$. The smallest number of colors of all RAC in graph G is RACN in graph G, symbolized by $rac(G)$ (Dafik et al., 2020).

Research results by Dafik et al. (2020) found a Theorem on the antimagic number relationship of tree graphs.

Theorem 1.1 Let T_n is tree of order n at least 3, $rac\ (T_n) = n - 1$ (Dafik et al., 2020).

Based on **Theorem 1.1** Brian et al. (2023) found results on the graph $\mathcal{F}_n \rhd T_m$.

Lemma 1.2 Let $\mathcal{F}_n \rhd T_m$ be a comb product of friendship graph $\mathcal{F}_n$ and T_m be any tree of order $m \geq 2$. The lower bound of $rac(\mathcal{F}_n \rhd T_m) \geq (rac(T_m))(|V(\mathcal{F}_n)|)$ (Septory et al., 2023).

RAC has the results of special graphs, which can be found at Budi et al. (2021), Dafik et al. (2020), Sulistiyono et al. (2020), and Septory et al. (2021). The result of RACN of graph operation can be found at Joedo et al. (2022), Septory et al. (2023), Septory et al. (2023), and Septory et al. (2023).

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METHODS

This study's method is direct proof, without relying on contradiction or opposing assumptions. The research process includes labeling the vertices in the comb operation graph $G \trianglerighteq H$, determining the lower and upper bounds of the rainbow antimagic connection number, constructing vertex labels, calculating edge weights, and checking the existence of rainbow paths between vertices. It also involves counting distinct edge weights, determining the upper bound based on these counts, and calculating $rac(G \trianglerighteq H)$ using the established bounds. Septory et al. (2023) also applied this approach in their study.

RESULT AND DISCUSSION

This section presents the RACN results for the comb product of the cycle graph C_3 with the path P_n and the broom graph $Br_{n,m}$ symbolized as $C_3 \triangleright P_n$ and $C_3 \triangleright Br_{n,m}$, respectively. In $C_3 \triangleright P_n$, the grafted vertices are x and v, wherein $deg(x) = 1$ and $d(v,x) = 1$. In $C_3 \triangleright Br_{n,m}$, the grafted vertices are x and v along a path of the broom graph, with $deg(x) = 1$ and $d(v,x) = 1$.

Theorem 2.1 For every $n \geq 3$, then $rac(C_3 \triangleright P_n) = 3n - 3$.

Proof. Graph $C_3 \triangleright P_n$ is a comb product of cycle C_3 and path graph P_n where $n \geq 3$, with the grafted vertices being vertex x and v where $deg(x) = 1$ and $d(v,x) = 1$. To prove this theorem, we divided it into 2 cases:

Case 1. Graph $C_3 \triangleright P_n$, wherein n at least 3 with grafted vertices being vertex x wherein $deg(x) = 1$.It means by the definition of the comb product of graph, we obtain a graph $C_3 \triangleright P_n$ with vertex set $V(C_3 \triangleright P_n) = \{x_j; 1 \leq j \leq 3\} \cup \{x_{j,i};\ 1 \leq i \leq n-1; 1 \leq j \leq 3\}$ and edge set $E(C_3 \triangleright P_n) = \{x_j x_{j+1}; 1 \leq j \leq 2\} \cup \{x_3 x_1\} \cup \{x_j x_{i,1};\ 1 \leq j \leq 3\} \cup \{x_{j,i} x_{j,i+1};\ 1 \leq i \leq n-2; 1 \leq j \leq 3\}$. The order of $C_3 \triangleright P_n$ is $|V(C_3 \triangleright P_n)| = 3 + 3(n-1)$ and the size of $C_3 \triangleright P_n$ is $|E(C_3 \triangleright P_n)| = 3 + 3(n-1)$. After identifying $V(C_3 \triangleright P_n)$ and $E(C_3 \triangleright P_n)$, we proceed to prove the RACN of $C_3 \triangleright P_n$. To begin, we must establish the lower bound of $rac(C_3 \triangleright P_n)$. **Theorem 1.1** explains that $rac(T_n) = n - 1$ it's mean that each copy of the path graph P_n in $C_3 \triangleright P_n$ must have a unique colors. Therefore, the lower bound for $rac(C_3 \triangleright P_n)$ is the number of colors $|V(C_3)|$ of the copies of the graph P_n. According to **Theorem 1.1**, wherein $rac(P_n) = n - 1$, and based on **Lemma 1.2** we have the same condition with the graph $C_3 \triangleright P_n$, then it follows that $rac(C_3 \triangleright P_n) \geq 3n -$

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3.

Next, we need to present the upper bound of $rac(C_3 \rhd P_n)$. Define the vertex labeling $f : V(C_3 \rhd P_n) \to \{1,2,\ldots,3n\}$ as described below.

$$f(x_1) = 2 \qquad f(x_{21}) = 3$$
$$f(x_2) = 4 \qquad f(x_{31}) = 1$$
$$f(x_3) = 5 \qquad f(x_{1i}) = 3i + 1, \text{for } 3 \le i \le n-1$$
$$f(x_{11}) = 7 \qquad f(x_{2i}) = 3i + 2, \text{for } 2 \le i \le n-1$$
$$f(x_{12}) = 6 \qquad f(x_{3i}) = 3i + 3, \text{for } 2 \le i \le n-1$$

The edge weights corresponding to the vertex labeling f can be expressed as follows:

$$w(x_1x_2) = 6 \qquad w(x_{12}x_{13}) = 16$$
$$w(x_2x_3) = 9 \qquad w(x_{21}x_{22}) = 12$$
$$w(x_3x_1) = 7 \qquad w(x_{31}x_{32}) = 10$$
$$w(x_1x_{11}) = 9 \qquad w(x_{1i}x_{1i+1}) = 6i + 5, \text{for } 3 \le i \le n-2$$
$$w(x_2x_{21}) = 7 \qquad w(x_{2i}x_{2i+1}) = 6i + 7, \text{for } 2 \le i \le n-2$$
$$w(x_3x_{31}) = 6 \qquad w(x_{3i}x_{3i+1}) = 6i + 9, \text{for } 2 \le i \le n-2$$
$$w(x_{11}x_{12}) = 13$$

The **Theorem 1.1**, presents that $rac(P_n) = n - 1$; since the edge set $|E(P_n)| = n - 1$, then the weight of each edge in the graph P_n is not the same. So the number of weights on the graph P_n is $n - 1$. According to the explanation above, we can determine the distinct weights of the graph $C_3 \rhd P_n$ is the number of colors $|V(C_3)|$ of copies of the graph P_n, then we have $3n - 3$.

This presents that the edge weight of $f: V(C_3 \rhd P_n) \to \{1,2,\ldots,3n\}$ induce an RAC of $3n - 3$ colors. Thus, $rac(C_3 \rhd P_n) \le 3n - 3$. The evaluation aims to demonstrate the existence of a rainbow $u - v$ path in $C_3 \rhd P_n$. According to the lower bound and upper bound, the precise value of $rac(C_3 \rhd P_n) = 3n - 3$. By utilizing the definition of the graph $C_3 \rhd P_n$ and assessing the rainbow $u - v$ path in the graph P_n, we can then evaluate the rainbow $u - v$ path in the graph $C_3 \rhd P_n$. According to **Theorem 1.1**, we have $rac(P_n) = n - 1$ and $E(P_n) = n - 1$, thus, there is a rainbow $u - v$ path for every $u, v \in V(P_n)$ with the condition of one color on i the copy path may be used on the edge in cycle C_3 which is not incident to vertex i. Thus, by the explanation, it is evident that a rainbow $u - v$ path exists for every $u, v \in V(C_3 \rhd P_n)$.

Figure 1 shows an illustration of the RAC of $C_3 \rhd P_n$ wherein n at least 3, with the grafted vertex x having $deg(x) = 1$.

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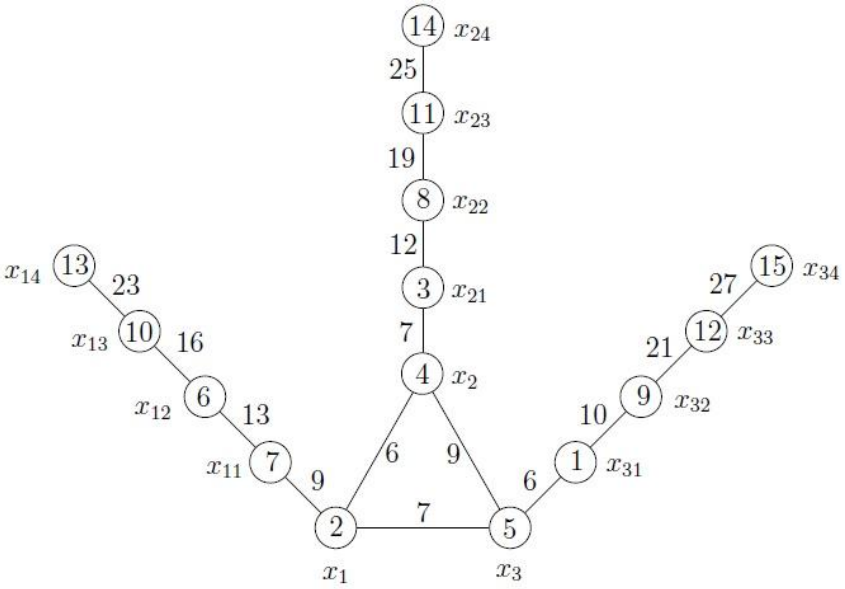


Figure 1. An illustration of RAC for the graph $C_3 \rhd P_5$

In Figure 1, the rainbow antimagic labeling is presented for the graph resulting from the comb operation $C_3 \rhd P_5$This labeling shows that the generated colors satisfy the requirements of RAC, meaning that every two vertices have a rainbow path.

Case 2. Graph $C_3 \rhd P_n$, wherein n at least 3 with grafted vertices being vertex u wherein $deg(x) = 1$, there is a vertex u wherein $d(u, x) = 1$.

In accordance with the definition of the comb product of graph, we obtain a graph $C_3 \rhd P_n$ with the vertex set $V(C_3 \rhd P_n) = \{x_j, y_j; 1 \le j \le 3\} \cup \{x_{j,i};\ 1 \le i \le n-2; 1 \le j \le 3\}$ and $E(C_3 \rhd P_n) = \{x_j x_{j+1}; 1 \le j \le 2\} \cup \{x_3 x_1\} \cup \{x_j y_j; 1 \le j \le 3\} \cup \{x_j x_{j,1}; 1 \le j \le 3\} \cup \{x_{j,i} x_{j,i+1}; 1 \le i \le n-2; 1 \le j \le 3\}$. The order of $C_3 \rhd P_n$ is $|V(C_3 \rhd P_n)| = 6 + 3(n-2)$ and the size of $C_3 \rhd P_n$ is $|E(C_3 \rhd P_n)| = 6 + 3(n-2)$.

After identifying $V(C_3 \rhd P_n)$ and $E(C_3 \rhd P_n)$, we proceed to prove the RACN of $rac(C_3 \rhd P_n)$. To begin, we must establish the lower bound of $rac(C_3 \rhd P_n)$. **Theorem 1.1** explains that $rac(T_n) = n - 1$ it's mean that each copy of the path graph P_n in $C_3 \rhd P_n$ must have a unique colors. Therefore, the lower bound for $rac(C_3 \rhd P_n)$ is the number of colors $|V(C_3)|$ of the copies of the graph P_n. According to **Theorem 1.1**, wherein $rac(P_n) = n - 1$ and based on **Lemma 1.2** we have the same condition with the graph $C_3 \rhd P_n$, then it follows that $rac(C_3 \rhd P_n) \ge 3n - 3$.

Next, we need to present the upper bound of $rac(C_3 \rhd P_n)$. Define the vertex labeling $f : V(C_3 \rhd P_n) \to \{1,2,\dots,3n\}$ as described below.

$$f(x_1) = 2 \qquad f(x_2) = 4$$

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$$f(x_3) = 5 \qquad f(x_{11}) = 6$$
$$f(y_1) = 7 \qquad f(x_{1i}) = 3i + 4, \text{for } 2 \le i \le n-2$$
$$f(y_2) = 3 \qquad f(x_{2i}) = 3i + 5, \text{for } 1 \le i \le n-2$$
$$f(y_3) = 1 \qquad f(x_{3i}) = 3i + 6, \text{for } 1 \le i \le n-2$$

The edge weights corresponding to the vertex labeling f can be expressed as follows:

$$w(x_1x_2) = 6 \qquad w(x_2x_{21}) = 12$$
$$w(x_2x_3) = 9 \qquad w(x_3x_{31}) = 14$$
$$w(x_3x_1) = 7 \qquad w(x_{11}x_{12}) = 16$$
$$w(x_1y_1) = 9 \qquad w(x_{1i}x_{1i+1}) = 6i + 11, \text{for } 3 \le i \le n-3$$
$$w(x_2y_2) = 7 \qquad w(x_{2i}x_{2i+1}) = 6i + 13, \text{for } 2 \le i \le n-3$$
$$w(x_3y_3) = 6 \qquad w(x_{3i}x_{3i+1}) = 6i + 15, \text{for } 2 \le i \le n-3$$
$$w(x_1x_{11}) = 8$$

The **Theorem 1.1**, presents that $rac(P_n) = n - 1$; since the edge set $|E(P_n)| = n - 1$, then the weight of each edge in the graph P_n is not the same. So the number of weights on the graph symbolized P_n is $n - 1$. According to the explanation above, we can determine the distinct weights of the graph $C_3 \rhd P_n$ is the number of colors $|V(C_3)|$ of copies of the graph P_n, then we have $3n - 3$.

This presents that the edge weight of $f: V(C_3 \rhd P_n) \to \{1,2,\ldots,3n\}$ induce a RAC of $3n - 3$ colors. Thus, $rac(C_3 \rhd P_n) \le 3n - 3$. The evaluation aims to demonstrate the existence of a rainbow $u - v$ path in $C_3 \rhd P_n$. By utilizing the definition of the graph $C_3 \rhd P_n$ and assessing the rainbow $u - v$ path in the graph P_n, we can then evaluate the rainbow $u - v$ path in the graph $C_3 \rhd P_n$. According to **Theorem 1.1**, we have $rac(P_n) = n - 1$ and $E(P_n) = n - 1$, thus, there is a rainbow $u - v$ path for every $u, v \in V(P_n)$ with the condition of one color on i the copy path may be used on the edge in cycle C_3 which is not incident to vertex i. Thus, By the explanation, it is evident that a rainbow $u - v$ path exists for every $u, v \in V(C_3 \rhd P_n)$. Following the lower bound and upper bound, the precise value of $rac(C_3 \rhd P_n) = 3n - 3$. □

Figure 2 shows an illustration of the RAC of $C_3 \rhd P_n$ wherein n at least 3, with the grafted vertex u having $deg(x) = 1$, there is a vertex u wherein $d(u, x) = 1$.

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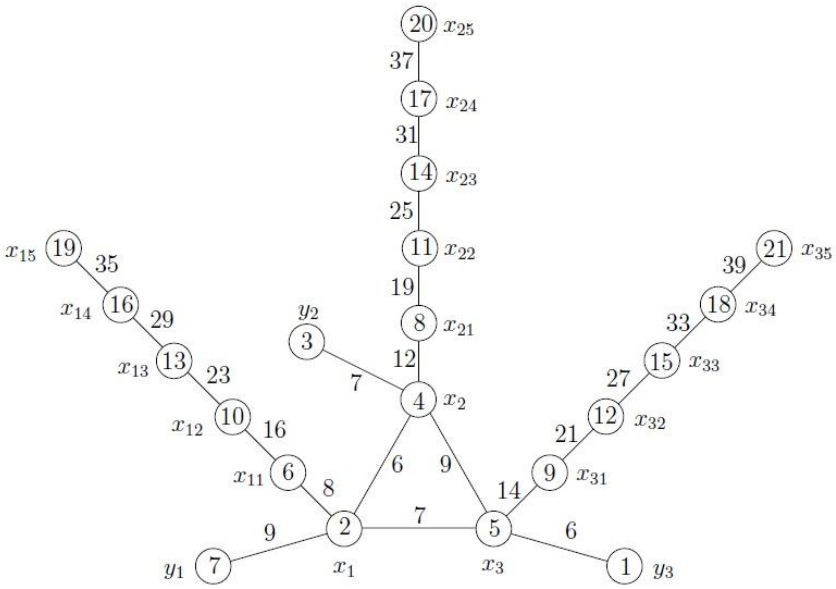


Figure 2. An illustration of RAC of graph $C_3 \rhd P_7$

In Figure 2, the rainbow antimagic labeling is presented for the graph resulting from the comb operation $C_3 \rhd P_7$This labeling shows that the generated colors satisfy the requirements of RAC, meaning that every two vertices have a rainbow path.

Theorem 2.2 For every $n \geq 4$ and $m \geq 2$, then $rac(C_3 \rhd P_n) = 3n + 3m - 3$.

Proof. Graph $C_3 \rhd P_n$ is a comb product of cycle C_3 and path graph P_n where $n \geq 3$, with the grafted vertices being vertex x and v where $deg(x) = 1$ and $d(v, x) = 1$. To prove this theorem, we divided it into 2 cases:

Case 1. Graph $C_3 \rhd Br_{n,m}$, wherein n at least 4 and m at least 2 with grafted vertices being vertex x on a path of Broom $Br_{n,m}$ wherein $deg(x) = 1$.

In accordance with the definition of the comb product of graph, we obtain a graph $C_3 \rhd Br_{n,m}$ with $V(C_3 \rhd Br_{n,m}) = \{x_k; 1 \leq k \leq 3\} \cup \{x_{k,i};\ 1 \leq i \leq n-1; 1 \leq k \leq 3\} \cup \{y_{k,j}; 1 \leq j \leq m; 1 \leq k \leq 3\}$ and $E(C_3 \rhd Br_{n,m}) = \{x_k x_{k+1}; 1 \leq k \leq 2\} \cup \{x_3 x_1\} \cup \{x_k x_{k,1}; 1 \leq k \leq 3\} \cup \{x_{k,i} x_{k,i+1};\ 1 \leq i \leq n-2; 1 \leq k \leq 3\} \cup \{x_{k,n-1} y_{k,j};\ 1 \leq j \leq m; 1 \leq k \leq 3\}$. The order of $C_3 \rhd Br_{n,m}$ is $|V(C_3 \rhd Br_{n,m})| = 3 + 3(n-1) + 3m$ and the size of $C_3 \rhd Br_{n,m}$ is $|E(C_3 \rhd Br_{n,m})| = 6 + 3(n-2) + 3m$.

After identifying $V(C_3 \rhd Br_{n,m})$ and $V(C_3 \rhd Br_{n,m})$, we proceed to prove the RACN of $rac(C_3 \rhd Br_{n,m})$. To begin, we must establish the lower bound of $rac(C_3 \rhd Br_{n,m})$. **Theorem 1.1** explains that $rac(Br_{n,m}) = n - 1$ it's mean that each copy of the path graph $Br_{n,m}$ in $C_3 \rhd Br_{n,m}$ must have a unique colors. Therefore, the lower bound for

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$rac(C_3 \rhd Br_{n,m})$ is the number of colors $|V(C_3)|$ of the copies of the graph $Br_{n,m}$. According to **Theorem 1.1**, wherein $rac(Br_{n,m}) = n + m - 1$ and based on **Lemma 1.2** we have the same condition with the graph $C_3 \rhd Br_{n,m}$, then it follows that $rac(C_3 \rhd Br_{n,m}) \geq 3n + 3m - 3$.

Next, we need to present the upper bound of $rac(C_3 \rhd Br_{n,m})$. Define the vertex labeling $f : V(C_3 \rhd Br_{n,m}) \to \{1,2,\ldots,3n + 3m\}$ as described below.

$$f(x_1) = 2$$
$$f(x_2) = 4$$
$$f(x_3) = 5$$
$$f(x_{11}) = 7$$
$$f(x_{12}) = 6$$
$$f(x_{21}) = 3$$
$$f(x_{31}) = 1$$
$$f(x_{1i}) = 3i + 1, \text{for } 3 \leq i \leq n-1$$
$$f(x_{2i}) = 3i + 2, \text{for } 2 \leq i \leq n-1$$
$$f(x_{3i}) = 3i + 3, \text{for } 2 \leq i \leq n-1$$
$$f(y_{1j}) = 3n + j, \text{for } 1 \leq j \leq m$$
$$f(y_{2j}) = 3n + m + j, \text{for } 1 \leq j \leq m$$
$$f(y_{3j}) = 3n + 2m + j, \text{for } 1 \leq j \leq m$$

The edge weights corresponding to the vertex labeling f can be expressed as follows:

$$w(x_1x_2) = 6$$
$$w(x_2x_3) = 9$$
$$w(x_3x_1) = 7$$
$$w(x_1x_{11}) = 9$$
$$w(x_2x_{21}) = 7$$
$$w(x_3x_{31}) = 6$$
$$w(x_{11}x_{12}) = 13$$
$$w(x_{12}x_{13}) = 16$$
$$w(x_{21}x_{22}) = 12$$
$$w(x_{31}x_{32}) = 10$$
$$w(x_{1i}x_{1i+1}) = 6i + 5, \text{for } 3 \leq i \leq n-2$$
$$w(x_{2i}x_{2i+1}) = 6i + 7, \text{for } 2 \leq i \leq n-2$$
$$w(x_{3i}x_{3i+1}) = 6i + 9, \text{for } 2 \leq i \leq n-2$$
$$w(x_{1n-1}y_{1j}) = 6n + j - 2, \text{for } 1 \leq j \leq m$$
$$w(x_{2n-1}y_{2j}) = 6n + m + j - 1, \text{for } 1 \leq j \leq m$$
$$w(x_{3n-1}y_{3j}) = 6n + 2m + j, \text{for } 1 \leq j \leq m$$

FollowingfollowingBy the **Theorem 1.1** presents that $rac(Br_{n,m}) = n + m - 1$; since the edge set $|E(Br_{n,m})| = n + m - 1$, then the weight of each edge in the graph $Br_{n,m}$ is not the same. So the number of weights on the graph $Br_{n,m}$ is $n + m - 1$. According to the explanation above, we can determine the distinct weights of the graph $C_3 \rhd Br_{n,m}$ is the number of colors $|V(C_3)|$ of copies of the graph $Br_{n,m}$, then we have $3n - 3m - 3$.This presents that the edge weights of $f: V(C_3 \rhd Br_{n,m}) \to \{1,2,\ldots,3n + 3m\}$ induce an RAC of $3n + 3m - 3$ colors. Thus, $rac(C_3 \rhd Br_{n,m}) \leq 3n + 3m - 3$. The evaluation aims to demonstrate the existence of a rainbow $u - v$ path in $C_3 \rhd Br_{n,m}$. By utilizing the definition of the graph $C_3 \rhd Br_{n,m}$ and assessing the rainbow $u - v$ path in the graph $Br_{n,m}$, we can then evaluate the rainbow $u - v$ path in the graph $C_3 \rhd Br_{n,m}$. According to **Theorem 1.1**, we have $rac(Br_{n,m}) = n + m - 1$ and $E(Br_{n,m}) = n + m -$

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1, thus, there is a rainbow $u-v$ path for every $u, v \in V(Br_{n,m})$ with the condition of one color on i the copy path may be used on the edge in cycle C_3 which is not incident to vertex i. Thus, in accordance with the explanation, it is evident that a rainbow $u-v$ path exists for every $u, v \in V(C_3 \rhd Br_{n,m})$. In accordance with the lower bound and upper bound, we have the precise value of $rac(C_3 \rhd Br_{n,m}) = 3n + 3m - 3$. □

Case 2. Graph $C_3 \rhd Br_{n,m}$, wherein n at least 4 and m at least 2 with grafted vertices being vertex x on a path of Broom $Br_{n,m}$ wherein $deg(x) = 1$.

In accordance with the definition of the comb product of graph, we obtain a graph $C_3 \rhd Br_{n,m}$ with $V(C_3 \rhd Br_{n,m}) = \{x_k; 1 \le k \le 3\} \cup \{x_{k,i};\ 1 \le i \le n-1; 1 \le k \le 3\} \cup \{y_{k,j};\ 1 \le j \le m; 1 \le k \le 3\}$ and $E(C_3 \rhd Br_{n,m}) = \{x_k x_{k+1}; 1 \le k \le 2\} \cup \{x_3 x_1\} \cup \{x_k x_{k,1}; 1 \le k \le 3\} \cup \{x_{k,i} x_{k,i+1}; 1 \le i \le n-2; 1 \le k \le 3\} \cup \{x_{k,n-1} y_{k,j};\ 1 \le j \le m; 1 \le k \le 3\}$. The order of $C_3 \rhd Br_{n,m}$ is $|V(C_3 \rhd Br_{n,m})| = 3 + 3(n-1) + 3m$ and the size of $C_3 \rhd Br_{n,m}$ is $|E(C_3 \rhd Br_{n,m})| = 6 + 3(n-1) + 3m$.

After identifying $V(C_3 \rhd Br_{n,m})$ and $V(C_3 \rhd Br_{n,m})$, we proceed to prove the RACN of $rac(C_3 \rhd Br_{n,m})$. To begin, we must establish the lower bound of $rac(C_3 \rhd Br_{n,m})$. **Theorem 1.1** explains that $rac(Br_{n,m}) = n - 1$ it's mean that each copy of the path graph $Br_{n,m}$ in $C_3 \rhd Br_{n,m}$ must have a unique colors. Therefore, the lower bound for $rac(C_3 \rhd Br_{n,m})$ is the number of colors $|V(C_3)|$ of the copies of the graph $Br_{n,m}$. According to **Theorem 1.1**, wherein $rac(Br_{n,m}) = n + m - 1$ and based on **Lemma 1.2** we have the same condition with the graph $C_3 \rhd Br_{n,m}$, then it follows that $rac(C_3 \rhd Br_{n,m}) \ge 3n + 3m - 3$.

Next, we need to present the upper bound of $rac(C_3 \rhd Br_{n,m})$. Define the vertex labeling $f : V(C_3 \rhd Br_{n,m}) \to \{1,2,\dots,3n+3m\}$ as described below.

$$f(x_1) = 2 \qquad f(x_{11}) = 6$$
$$f(x_2) = 4 \qquad f(x_{1i}) = 3i + 4, \text{for } 2 \le i \le n-2$$
$$f(x_3) = 5 \qquad f(x_{2i}) = 3i + 5, \text{for } 1 \le i \le n-2$$
$$f(y_1) = 7 \qquad f(x_{3i}) = 3i + 6, \text{for } 1 \le i \le n-2$$
$$f(y_2) = 3 \qquad f(z_{1j}) = 3n + j, \text{for } 1 \le j \le m$$
$$f(y_3) = 1$$

The edge weights corresponding to the vertex labeling f can be expressed as follows:

$$w(x_1x_2) = 6$$
$$w(x_2x_3) = 9$$

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$$w(x_3x_1) = 7$$
$$w(x_1y_1) = 9$$
$$w(x_2y_2) = 7$$
$$w(x_3y_3) = 6$$
$$w(x_1x_{11}) = 8$$
$$w(x_2x_{21}) = 12$$
$$w(x_3x_{31}) = 14$$
$$w(x_{11}x_{12}) = 16$$
$$w(x_{1i}x_{1i+1}) = 6i + 11, \text{for } 2 \le i \le n-3$$
$$w(x_{2i}x_{2i+1}) = 6i + 13, \text{for } 1 \le i \le n-3$$
$$w(x_{3i}x_{3i+1}) = 6i + 15, \text{for } 1 \le i \le n-3$$
$$w(x_{1n-2}z_{1j}) = 6n + j - 2, \text{for } 1 \le j \le m$$
$$w(x_{2n-2}z_{2j}) = 6n + m + j - 1, \text{for } 1 \le j \le m$$
$$w(x_{3n-2}z_{3j}) = 6n + 2m + j, \text{for } 1 \le j \le m$$

The **Theorem 1.1** presents that $rac(Br_{n,m}) = n + m - 1$; since the edge set $|E(Br_{n,m})| = n + m - 1$, then the weight of each edge in the graph $Br_{n,m}$ is not the same. So the number of weights on the graph $Br_{n,m}$ is $n + m - 1$. According to the explanation above, we can determine the distinct weights of the graph $C_3 \rhd Br_{n,m}$ is the number of colors $|V(C_3)|$ of copies of the graph $Br_{n,m}$, then we have $3n - 3m - 3$.

This presents that the edge weights of $f: V(C_3 \rhd Br_{n,m}) \to \{1,2,\ldots,3n + 3m\}$ induce an RAC of $3n + 3m - 3$ colors. Thus, $rac(C_3 \rhd Br_{n,m}) \le 3n + 3m - 3$. The evaluation aims to demonstrate the existence of a rainbow $u - v$ path in $C_3 \rhd Br_{n,m}$. By utilizing the definition of the graph $C_3 \rhd Br_{n,m}$ and assessing the rainbow $u - v$ path in the graph $Br_{n,m}$, we can then evaluate the rainbow $u - v$ path in the graph $C_3 \rhd Br_{n,m}$. According to **Theorem 1.1**, we have $rac(Br_{n,m}) = n + m - 1$ and $E(Br_{n,m}) = n + m - 1$, thus, there is a rainbow $u - v$ path for every $u, v \in V(Br_{n,m})$ with the condition of one color on i the copy path may be used on the edge in a cycle C_3 which is not incident to the vertex i. Thus, in light of the explanation, it is evident that a rainbow $u - v$ path exists for every $u, v \in V(C_3 \rhd Br_{n,m})$. In light of the lower and upper bounds, we have the precise value of $rac(C_3 \rhd Br_{n,m}) = 3n + 3m - 3$. □

CONCLUSION

By the result in the paper the precise value of the RACN of graph $C_3 \rhd P_n$ wherein n at least 3, with the grafted vertices being vertex x and v wherein $deg(x) = 1$ and

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$d(v,x) = 1$ has the same rainbow antimagic connection number, which is $rac(C_3 \rhd P_n) = 3n - 3$. Wherein n at least 4 and m at least 2, with the grafted vertices being vertex x and u on a path of broom $Br_{n,m}$ wherein $deg(x) = 1$ and $d(u,x) = 1$ has the same rainbow antimagic connection number, which is $rac(C_3 \rhd Br_{n,m}) = 3n + 3m - 3$. Therefore, this study found results that can be developed by raising open problems.

OPEN PROBLEM 1. Analyze the RACN of graph $G \rhd T_n$ wherein G is not a cycle graph C_3.

OPEN PROBLEM 2. Analyze the RACN of graph$C_3 \rhd H$, wherein H is non-tree graph.

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AUTHOR CONTRIBUTIONS

Brian Juned Septory: Conceptualization, original draft writing, editing, and visualization;

Dwi Agustin Retnowardani: Review and editing of the writing, formal analysis, and methodology;

Kamal Dliou: Review and supervision.

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